The effect of curvature on convexity properties of harmonic functions and eigenfunctions

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Dedicated to Shmuel Agmon with admiration and gratitude on the occasion of his 90th birthday

Abstract

We give a proof of the growth bound of Laplace-Beltrami eigenfunctions due to Donnelly and Fefferman which is probably the easiest and the most elementary one. Our proof also gives new quantitative geometric estimates in terms of curvature bounds which improve and simplify previous work by Garofalo and Lin. The proof is based on a generalization of a convexity property of harmonic functions in $\mathbb{R}^n$ to harmonic functions on Riemannian manifolds following Agmon's ideas.

1 Introduction

In their seminal paper [DF88] Donnelly and Fefferman found growth bounds (DF-growth bound) for eigenfunctions on compact Riemannian manifolds. Roughly, they showed that a λ-eigenfunction grows like a polynomial of order $\sqrt{\lambda}$ at most. This result is central in the study of eigenfunctions. In [DF88] it was applied to prove Yau's conjecture on real analytic manifolds. Namely, sharp upper and lower bounds on the size of the nodal set on real analytic manifolds were found. The proof of the growth bound in [DF88] went through a fine version of a Carleman type inequality for the operator $\Delta + \lambda$, with a careful geometric choice of the weight function.

Recently after, Lin ([Lin91]), based on an earlier work with Garofalo ([GL86]), gave a simpler proof of the growth bound. This proof is based

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on properties of the spherical L^2-norm, $q(r)$ (defined in (2.2)), for harmonic functions. It had been known ([Agm66, Alm79]) that in $\mathbb{R}^n$, $\log q$ is monotonically increasing and convex as a function of $\log r$. Equivalently, $rq'(r)/q(r)$ is monotonically increasing. Garofalo-Lin showed that for a harmonic function defined on a general Riemannian manifold $\mathrm{e}^{\Lambda r} rq'/q$ is monotonically increasing in (0, R), where Λ and R are some positive constants depending on bounds on the Riemannian metric, on its first derivatives and on the ellipticity constant of the Riemannian metric. This result can be viewed as an approximated convexity result. The proof of this result was based on a non-trivial geometric variational argument which was first used by Almgren [Alm79].

The first aim of this paper is to give new geometric estimates on Λ and R in terms of the curvature of the manifold. Namely, we find that all one needs is a lower and an upper bound on the sectional curvature in order to guarantee the existence of Λ and R. Moreover, we show that in fact $\mathrm{e}^{C_1 r^2 K} rq'(r)/q(r)$ is monotonic in $(0, R)$, where K is an upper bound on the curvature, R is the minimum of $C_2/\sqrt{K^+}$ and the injectivity radius, and C_1, C_2 depend only on the dimension of the manifold. We emphasize that our result distinguishes between negative and positive curvatures. This is the content of the main Theorem 2.3.

The second aim of this paper is to have a simple proof of the DF-growth bound for eigenfunctions. Due to the importance of this result three simplifications to its proof had been previously given by different authors in the course of years, which we briefly survey:

The idea of Lin in [Lin91] was to consider a conic manifold, N, over M and to extend the eigenfunction u_λ to a harmonic function on N. Then, Lin applied the monotonicity property of $\mathrm{e}^{\Lambda r} rq'/q$ from [GL86] for the harmonic function obtained, and went back to the eigenfunction.

Jerison and Lebeau applied in [JL99] a similar extension of eigenfunctions. Then, they could use standard Carleman type inequalities for harmonic functions, instead of the original approach taken by Donnelly and Fefferman in which a special and delicate Carleman type inequality for eigenfunctions was used.

In dimension two Nazarov-Polterovich-Sodin [NPS05] took advantage of the conformal coordinates, thus letting them to simplify the problem by considering only the standard Laplace operator in $\mathbb{R}^2$. Then, they extend the eigenfunction to a harmonic function on $N = M \times \mathbb{R}$, and apply convexity argument on the harmonic function (in $\mathbb{R}^3$) obtained. Their proof of con-

vexity of $\log q$ is considerably simpler than the variational approach taken in [GL86]. It is close in spirit to Agmon's approach. This gives the easiest proof of the DF-growth bound in dimension two, since no need for variational arguments or Carleman type inequalities at all is required.

This paper extends the work started in [NPS05], to dimensions ≥ 3, where no conformal coordinates exist. We follow and generalize Agmon's ideas in [Agm66], where a general approximated convexity theorem for second order elliptic equations is proved by considering them as an abstract second order ODE. Our contribution here comes in adding the geometric point of view, clarifying the way curvature affects the Euclidean result. Our proof also simplifies and improves Agmon's results in [Agm66]. In this way we are able circumvent the need to use the non-trivial variational argument in [GL86] or any Carleman type inequality.

Organization of the paper. The main result is presented in section 2. In section 3 we recall a way eigenfunctions can be extended to harmonic functions and the translation of the convexity property of harmonic functions to a local growth bound on eigenfunctions. In section 4 we conclude the proof of the DF-growth bound on compact manifolds. and we outline the proof of Yau's conjecture in [DF88]. Sections 3 and 4 are strongly based on [NPS05]. In section 5 we give the proof of the main theorem. In section 6 we consider constant curvature manifolds as examples to the main theorem and find a second proof in some of these cases. In section 7 we discuss several open questions.

Notation. Throughout this paper $C_i, C_i(n)$ denote positive constants which depend only on dimension. The positive constants $C_g(\ldots)$ depend on bounds on the metric g, its first derivatives, its ellipticity constant and additional parameters appearing in parentheses.

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2 Main Theorem: A perturbed log-convexity property of harmonic functions

Let u be a harmonic function in $\mathbb{R}^n$. Let $q(r)$ denote the square of the spherical L^2-norm:

$$q(r) := \int_{S_r} u^2 \, d\sigma_r \ ,$$

where S_r denotes the sphere of radius r centered at 0, and $d\sigma_r$ is the standard area measure on S_r. It's easy to check that q is a convex function of $\log r$. It turns out that even $\log q$ is a convex function of $\log r$:

Theorem 2.1 ([Agm66]). *q has the following two properties:*

(i) $q'(r) \geq \dfrac{n-1}{r} q(r)$,

(ii) $q''(r) + \dfrac{1}{r} q'(r) - \dfrac{q'(r)^2}{q(r)} \geq 0.$

In dimension 2 this can be seen by a complex analysis argument. In higher dimensions this fact goes back at least to Agmon ([Agm66]), and it was rediscovered by Almgren [Alm79]. Landis ([Lan63, Ch. II.2]) found also several results close in spirit to that one. All these kinds of results were inspired by Hadamard's Three Circles Theorem (See [Ahl78, Ch. 6.2]), which shows log-convexity of the spherical L^∞-norm for a holomorphic function.

Remark. It is somewhat surprising that the fundamental solution does not play a role here: $\log q$ is a convex function of $\log r$ in *all* dimensions. The weaker statement is that $\log q$ is a convex function of $G(r) = -1/r^{n-2}$, which is equivalent to $q\Delta \log q = q''(r) + \frac{n-1}{r} q'(r) - \frac{q'(r)^2}{q(r)} \geq 0$.

When considering harmonic functions on manifolds, one expects a perturbed version of Theorem 2.1 in small geodesic balls. However, it is not clear a priori how far from the center this perturbation goes and how curvature controls it. Theorem 2.3 below will give an answer to these questions. Let u be a harmonic function defined in a small geodesic ball of a Riemannian manifold N. Let

$$q(r) := \int_{S(r)} u^2 \, dA_r \ , \tag{2.2}$$

where $S(r)$ is a geodesic sphere centred at $p \in N$, and dA_r is the area form on $S(r)$. $\sqrt{q}$ is the spherical L^2-norm on a geodesic sphere of radius r. We let Sec_N denote the sectional curvature of N, $K^+ = \max\{K, 0\}$,

$$\sin_K r = \begin{cases} \frac{\sin(r\sqrt{K})}{\sqrt{K}} & , \quad K > 0, \\ r & , \quad K = 0, \\ \frac{\sinh(r\sqrt{-K})}{\sqrt{-K}} & , \quad K < 0 \; . \end{cases}$$

and $\cot_K r = (\sin_K r)'/(\sin_K r)$. We can now state our main result:

Theorem 2.3. *Let N be a Riemannian manifold. Let u be a harmonic function on a geodesic ball in N, and q defined as in (2.2). Let $\kappa, K \in \mathbb{R}$, $\kappa \leq K$. Let $R = \min\left(\mathrm{inj}(M), \pi/(2\sqrt{K^+})\right)$. We have*

(i) If $\mathrm{Sec}_N \leq K$ then $q(r)/(\sin_K r)^{n-1}$ is monotonically increasing for $r < R$. Equivalently,

$$(\log q)'(r) \geq (n-1)(\cot_K r) \; .$$

(ii) If $\kappa \leq \mathrm{Sec}_N \leq K$ then for $r < R$

$$(\log q)''(r) + (\cot_K r)(\log q)'(r) + (n+1)(\cot_\kappa r - \cot_K r)(\log q)'(r) \geq -K - (n-2)K^+ - (2n-3)(K-\kappa) \; .$$

The proof of the theorem is given in Section 5.

Remarks:

- It looks like in dimensions $n \geq 3$ the result for negative curvature is better. However, this seems to be an artificial phenomenon since one could state part (ii) of the theorem with the function $\tilde{q} = q/(\sin_K r)^{n-1}$ replacing q: Then, the RHS becomes $-(n-2)K^- - (2n-3)(K-\kappa)$ which gives "advantage" to positive curvature in dimensions $n \geq 3$ (see also the discussion in Section 7.2).

- For the constant non-positive (non-negative) curvature case we get an exact convexity statement for $\log q$ (for $\log \tilde{q}$).

- Comparing to the result of Garofalo and Lin in [GL86], from part (ii) one deduces that $e^{6nr^2K} rq'(r)/q(r)$ is monotonically increasing for $r < R$. Observe that besides the explicit estimates of Λ and R mentioned in the introduction this gives also a correction of the result in [GL86] in the power of r in the exponential term. Moreover, the statement here is more geometric in nature.

We now would like to have an integrated version of Theorem 2.3. We restrict our attention only to the case $\kappa = -K$, $K > 0$. We obtain a local doubling estimate for harmonic functions (see proof in Section 5.5).

Corollary 2.4. *Let N be a complete Riemannian manifold of dimension n with $|\mathrm{Sec}_N| \leq K$. Then*

$$\frac{q(2r)}{q(r)} \leq \left(\frac{q(2s)}{q(s)}\right)^{1+32nr^2K}$$

for all $r < s < 1/(4\sqrt{nK})$.

3 Harmonic extension of Eigenfunctions

In this section we recall a connection between harmonic functions and eigenfunctions found in [Lin91, JL99, NPS05]. Let M be a Riemannian manifold of dimension m. Let u_λ be a λ-eigenfunction on M. Consider the direct product Riemannian manifold $N = M \times \mathbb{R}$ of dimension $n = m + 1$, where the metric on $\mathbb{R}$ is the standard one. Let H be the following function on N:

$$\forall x \in M, t \in \mathbb{R} \quad H(x,t) := u_\lambda(x)\cosh(\sqrt{\lambda}t) \ .$$

H extends u_λ to N and is harmonic on N, since the Laplacian on N can be written as

$$\Delta_N u = \Delta_M u + \frac{\partial^2 u}{\partial t^2} \ .$$

On N we take geodesic coordinates $(r, \theta_1, \ldots, \theta_{n-1})$ in a neighborhood of the point $(p, 0) \in N$. In these coordinates the metric g_N takes the following form

$$g_N = \ dr^2 + r^2 a_{ij}\, d\theta^i\, d\theta^j \quad 1 \leq i, j \leq n-1 \, .$$

We let $\hat\theta = (\theta_1, \ldots, \theta_{m-1})$, and $b_{ij}(r, \hat\theta) := a_{ij}(r, \hat\theta, 0)$.

$$g_M = dr^2 + r^2 b_{ij}\, d\theta^i\, d\theta^j, \quad 1 \le i, j \le m-1 \ .$$

Accordingly, the equation $\Delta_N H = 0$ can be written in these coordinates as

$$H_{rr} + \left(\frac{n-1}{r} + \gamma(r,\theta)\right) H_r + \frac{1}{r^2}\Delta_{S(r)} H = 0 \ ,$$

where $\gamma(r,\theta) = (\sqrt{a})_r/\sqrt{a}$ with $a = \det(a_{ij})$, and $\Delta_{S(r)}$ is the spherical Laplacian on the geodesic sphere of radius r:

$$\Delta_{S(r)} H := \frac{1}{\sqrt{a}} \frac{\partial}{\partial \theta^i}\left(\sqrt{a} a^{ij} \frac{\partial H}{\partial \theta^j}\right)$$

The following lemma relates $q(r)^{1/2}$, the spherical L^2-norm of the harmonic function H on an $(n-1)$-dimensional sphere of radius r, to $M_r(u_\lambda)$, the L^∞-norm of the eigenfunction u_λ on an $m = n-1$ dimensional *ball* of radius r. Let $M_r(u_\lambda) := \max_{B(p,r)} |u_\lambda(x)|$.

Lemma 3.1. *Suppose M is a complete Riemannian manifold with bounded geometry. Fix $0 < \alpha < 1$, $\varepsilon > 0$. Then for all $0 < r < \mathrm{inj}_M$,*

$$C_{\alpha,\varepsilon} r^m (1 + r\sqrt{\lambda})^{-m-\varepsilon} M_{\alpha r}(u_\lambda)^2 \le q(r) \le C_2 r^m \mathrm{e}^{2r\sqrt{\lambda}} M_r(u_\lambda)^2 \ .$$

where $C_{\alpha,\varepsilon}$ depends on α, ε and the metric, and C_2 depends on the metric.

Proof. Let us denote by $d\sigma(\hat\theta)$ the standard volume form on the unit sphere of dimension $m-1$.

$$\begin{aligned} q(r) = 2\int_0^r \int & u_\lambda(\rho,\hat\theta)^2 \cosh^2(\sqrt{\lambda}\sqrt{r^2-\rho^2}) \\ & \cdot \rho^{m-1} \sqrt{b(\rho,\hat\theta)} \frac{r}{\sqrt{r^2-\rho^2}}\ d\hat\theta\ d\rho \\ \le C M_r(u_\lambda)^2 & (3 + \mathrm{e}^{2r\sqrt{\lambda}}) \int_0^r \int_{S^{m-1}} \rho^{m-1} \frac{r}{\sqrt{r^2-\rho^2}}\, d\sigma(\hat\theta) d\rho \\ & = C\omega_m r^m M_r(u_\lambda)^2 (3 + \mathrm{e}^{2r\sqrt{\lambda}}) \ , \end{aligned}$$

where we used the fact that the volume element is bounded from above by the metric ([BC64, Ch. 11, Th. 15]).

On the other hand, we have

$$q(r) \geq 2 \int_0^r \int u_\lambda(\rho, \hat{\theta})^2 \rho^{m-1} \sqrt{b}\, d\hat{\theta}\, d\rho = \int_{B^m(p,r)} u_\lambda^2 \, d\mathrm{Vol}_M \ .$$

Hence, from elliptic regularity we get

$$q(r) \geq C_{\alpha,\varepsilon} M_{\alpha r}(u_\lambda)^2 r^m (1 + r\sqrt{\lambda})^{-m-\varepsilon} \ ,$$

where $C_{\alpha,\varepsilon}$ depends on the metric, on α and on ε. □

From Corollary 2.4 and Lemma 3.1 we find

Theorem 3.2. *Let M be a complete Riemannian manifold of dimension m with $|\mathrm{Sec}_M| \leq K$. Then for all $r \leq s < C/\sqrt{K}$*

$$\frac{M_{3r}(u_\lambda)}{M_{2r}(u_\lambda)} \leq C_1 \mathrm{e}^{C_2 s\sqrt{\lambda}} \left(\frac{M_{8s}(u_\lambda)}{M_{3s}(u_\lambda)} \right)^{1+C_3 r^2 K} \ ,$$

where the constants C_2, C_3 denote positive constants which depend only on the injectivity radius of M, while C_1 depends on bounds on the metric, its derivatives and its ellipticity constant.

Remark. The subindices $3r, 2r, 8s, 3s$ can be replaced by $\beta r, r, \gamma s, s$ respectively, where $1 < \beta < 2$ and $\gamma > \beta$. The constants C_2, C_3 can be taken to be independent of β, γ, while $C_1 \to \infty$ as $\gamma/\beta \to 1$.

4 Two global growth estimates

In this section we deduce from the local inequality in Theorem 3.2 two global results in the compact case.

4.1 Large values on large balls

Theorem 4.1. *Let M be a compact Riemannian manifold of dimension m. Then for all eigenfunctions u_λ and $r > 0$*

$$\frac{\max_{B(x,r)} |u_\lambda|}{\max_M |u_\lambda|} \geq C_g(r, d_M) \mathrm{e}^{-C_2 d_M \sqrt{\lambda}} \quad \forall x \in M \ ,$$

where d_M is the diameter of M,

Proof. Normalize u_λ so $\max_M |u_\lambda| = 1$. Take $r = s$ in Theorem 3.2. We get

$$M_{3r}(u_\lambda)^{2+C_3 r^2 K} \leq C_1 \mathrm{e}^{C_2 r\sqrt{\lambda}} M_{8r}(u_\lambda)^{1+C_3 r^2 K} M_{2r}(u_\lambda) \leq C_1 \mathrm{e}^{C_2 r\sqrt{\lambda}} M_{2r}(u_\lambda)\ . \tag{4.2}$$

Let $|u_\lambda(x_0)| = 1$. Fix $r_0 > 0$ small enough in order to apply Theorem 3.2. Take a point x in M. There exists a sequence of points $x_0, x_1, \ldots x_N = x$, such that $d(x_k, x_{k+1}) < r_0$, for $0 \leq k \leq N-1$, where N only depends on r_0 and the diameter of M. Inequality (4.2) gives

$$\max_{B(x_k,2r_0)} |u_\lambda| \geq C_1^{-1} \mathrm{e}^{-C_2 r_0\sqrt{\lambda}} \max_{B(x_k,3r_0)} |u_\lambda|^{2+C_2 r^2 K} \geq \\ C_1^{-1} \mathrm{e}^{-C_2 r_0\sqrt{\lambda}} \max_{B(x_{k-1},2r_0)} |u_\lambda|^3\ . \tag{4.3}$$

Multipliying the inequalities (4.3) for $1 \leq k \leq N$ gives

$$\max_{B(x,2r_0)} |u_\lambda| \geq C_1^{-N} \mathrm{e}^{-C_2 N r_0\sqrt{\lambda}} \geq C_1^{-N} \mathrm{e}^{-C_2 d\sqrt{\lambda}}\ .$$

□

4.2 Global DF growth Bound

Theorem 4.4 ([DF88]). *For all eigenfunctions u_λ, $x \in M$ and $r > 0$*

$$\frac{\max_{B(x,3r)} |u_\lambda|}{\max_{B(x,2r)} |u_\lambda|} \leq C_g(d_M) \mathrm{e}^{C_2 d_M \sqrt{\lambda}}\ .$$

Proof. Let $R > 0$ be as in Theorem 4.1. If $r \geq R$ the theorem follows from Theorem 4.1. Else, Theorems 3.2 and 4.1 tell us that

$$\frac{M_{3r}(u_\lambda)}{M_{2r}(u_\lambda)} \leq C_g \mathrm{e}^{C_2 R\sqrt{\lambda}} \left(\frac{M_{8R}(u_\lambda)}{M_{3R}(u_\lambda)}\right)^2 \leq C_g(d_M) \mathrm{e}^{2C_2 d_M \sqrt{\lambda}}\ .$$

□

4.3 Outline of the proof of Yau's Conjecture for real analytic manifolds

Yau's conjecture for C^∞ closed compact Riemannian manifolds is

Conjecture 4.5 ([Yau82]). *Let u_λ be a λ-eigenfunction on M. Then,*

$$C_1\sqrt{\lambda} \le \mathrm{Vol}_{n-1}(\{u_\lambda = 0\}) \le C_2\sqrt{\lambda}\ ,$$

where C_1, C_2 are constants independent of λ.

The conjecture was proved in the case of real analytic Riemannian metrics in [DF88]. A major ingredient of the proof was Theorem 4.4. We outline here the idea:

Lower bound. Let $B \subset M$ be a ball of radius $r = C/\sqrt{\lambda}$ such that u_λ vanishes at the center of B. One can cover, say, $1/2$ of the volume of M by a disjoint collection $\mathcal{B}$ of such balls ([Brü78]). One observes that if the growth of u_λ in a ball B is smaller than, say, 20 then one can control from below the size of the nodal set in B. This can be seen for harmonic functions in the unit ball using the mean value principle and the isoperimetric inequality, and can be adapted to eigenfunctions on balls of radius $C/\sqrt{\lambda}$.

The main claim is that on at least, say, 10% of the balls in the collection $\mathcal{B}$ the growth is bounded by 20.

We can assume M is contained in one coordinate neighbourhood $U = \{|x| < 30\} \subset \mathbb{R}^n$. One can continue the function u_λ to a holomorphic function F on $U \times U \subset \mathbb{C}^n$. We assume $F|_{U\times\{0\}} = u_\lambda$ and we set $Q \subset U \times \{0\}$ to be a Euclidean real cube. The point is that due to Theorem 4.4 the growth of F^2 in $U \times U$ is controlled by $\sqrt{\lambda}$.

We subdivide Q to small sub-cubes Q_ν of sides $1/\sqrt{\lambda}$. The next idea is that in order to bound the growth of F in a cube Q_ν by a constant independent of λ it is enough to say that F is close to its average on Q_ν for most of the points in Q_ν. This property behaves well under averaging. Therefore, it can be reduced to a dimension one problem: $Q = [-1, 1]$, $B = |z| < 2$, F is a holomorphic function defined on B, F is real on the real line. and its growth is bounded by $\sqrt{\lambda}$. First we replace F by a polynomial P of degree $\sqrt{\lambda}$. One divides Q into segments Q_ν of size $1/\sqrt{\lambda}$. One has to show that P is close to its average on 10% of these intervals. To that end the Hilbert transform is called.

Upper bound. The size of the nodal set is estimated from above by Crofton's formula. To estimate from above the number of zeros on a real line interval $I \subset Q$ one uses Jensen's formula in a complex line $\mathbb{C}$ containing I. For this one has to have a bound on the growth of F in $U \times U$.

5 Proof of Theorem 2.3

5.1 Preliminary geometric estimates

Let N be a Riemannian manifold of dimension n. Fix a point p, and let $r(x) = \mathrm{dist}(x, p)$. Let

$$\gamma_K = \Delta r - (n-1)\cot_K r \ .$$

γ_K is controlled by the curvature of N:

Lemma 5.1. *If $\kappa \le \mathrm{Sec}_N \le K$ then $0 \le \gamma_K \le (n-1)(\cot_\kappa r - \cot_K r)$.*

Proof. Both parts directly follow from the Hessian Comparison Theorem ([BC64, SY94]). ☐

Lemma 5.2. *Suppose $\kappa \le \mathrm{Sec}_N \le K$. Then, we have*

$$\gamma_{K,r} \ge -(n-1)(K-\kappa) \ .$$

Proof. We know ([Pet06, Ch. 9.1])

$$\gamma_{K,r} = (\Delta r)_r + \frac{n-1}{(\sin_K r)^2} = -\mathrm{Ric}(\partial_r, \partial_r) - \|\mathrm{Hess}(r)\|^2 + \frac{n-1}{(\sin_K r)^2} \ .$$

By the Hessian comparison theorem ([SY94])

$$(\cot_K r)\|X\|^2 \le \mathrm{Hess}(r)(X,X) \le (\cot_\kappa r)\|X\|^2 \ . \tag{5.3}$$

Hence,

$$|\mathrm{Hess}(r)(X,X)|^2 \le (\cot_\kappa r)^2 \|X\|^4 \ .$$

We can choose an orthonormal basis $(\partial_r, e_1, \ldots, e_{n-1})$ in which $\mathrm{Hess}(r)$ is diagonalized. Then we see

$$\|\mathrm{Hess}(r)\|^2 = \sum |\mathrm{Hess}(r)(e_i, e_i)|^2 \le (n-1)(\cot_\kappa r)^2 \ .$$

Consequently,

$$\begin{aligned} \gamma_{K,r} \ge -(n-1)K - (n-1)(\cot_\kappa r)^2 + \frac{n-1}{(\sin_K r)^2} &= (n-1)(\cot_K^2 r - \cot_\kappa^2 r) \\ &\ge -(n-1)(K-\kappa) \end{aligned}$$

where the last inequality follows from parts (iii) and (iv) of Lemma 5.4 below. ☐

Lemma 5.4. (i) $-1/3 \le (\sqrt{x}\cot\sqrt{x})' \le 0$ *for all* $0 \le x < (\pi/2)^2$.

(ii) $0 \le (\sqrt{x}\coth\sqrt{x})' \le 1/3$ *for all* $x \ge 0$.

(iii) $-1 \le (x\cot^2\sqrt{x})' \le 0$ *for all* $0 \le x < (\pi/2)^2$.

(iv) $0 \le \left(x\coth^2\sqrt{x}\right)' \le 1$ *for all* $x \ge 0$

Proof. We prove the right inequality in (ii): Since $y\coth y \ge 1$, we have $(3y + 2y\sinh^2 y)' \ge 3(\cosh y \sinh y)'$. Integrating, we conclude that

$$3y + 2y\sinh^2 y \ge 3\cosh y \sinh y \ .$$

Equivalently, $(y\coth y)' \le 2y/3$. Hence, $(\sqrt{x}\coth\sqrt{x})' \le 1/3$.

We prove the left inequality in (iii):

$$(x\cot^2\sqrt{x})' = \cot^2\sqrt{x} - \frac{\sqrt{x}\cot\sqrt{x}}{\sin^2\sqrt{x}} \ . \tag{5.5}$$

Observe that for $0 \le y < \pi/2$

$$y\cot y \le 1 \ . \tag{5.6}$$

From (5.5) and (5.6) it follows that

$$(x\cot^2\sqrt{x})' \ge \cot^2\sqrt{x} - \frac{1}{\sin^2\sqrt{x}} = -1 \ .$$

The proofs of the all other inequalities in the Lemma are omitted. $\square$

5.2 Choice of coordinates and notations

We take geodesic polar coordinates centred at $p \in N$. Fix any $K \in \mathbb{R}$. The metric can be written as

$$g = \ dr^2 + (\sin_K r)^2 (a_K)_{ij}\ d\theta^i\ d\theta^j \ ,$$

where θ^i are coordinates on the standard unit sphere $S^{n-1} \subset \mathbb{R}^n$.

We denote the determinant of the matrix $(a_K)_{ij}$ by a_K. The Laplacian on N can be written as

$$(\Delta f)(r,\theta) = f_{rr}(r,\theta) + ((n-1)\cot_K r + \gamma_K) f_r(r,\theta) + \frac{1}{(\sin_K r)^2}\left(\Delta_S f(r,\cdot)\right)(\theta) \ ,$$

where Δ_S is the following operator acting on functions g defined on S^{n-1}:

$$(\Delta_S g)(\theta) := \frac{1}{\sqrt{a_K}} \frac{\partial}{\partial \theta^i} \left(a_K^{ij} \sqrt{a_K} \frac{\partial g}{\partial \theta^j} \right) .$$

We emphasize that the definition of Δ_S depends on our choice of K. With these definitions we also have

$$\gamma_K = \frac{(\sqrt{a_K})_r}{\sqrt{a_K}} .$$

5.3 Proof of part (i)

We observe that

$$q(r) = \int u^2 (\sin_K r)^{n-1} \sqrt{a_K}\, d\theta ,$$

where the integration is understood to be performed over the parameter space $[0, \pi]^{n-2} \times [0, 2\pi]$ for S^{n-1} in $\mathbb{R}^{n-1}$. A straightforward computation shows

Lemma 5.7.

$$\begin{aligned} q'(r) = \int 2uu_r (\sin_K r)^{n-1} \sqrt{a_K}\, d\theta + \int u^2 \gamma_K (\sin_K r)^{n-1} \sqrt{a_K}\, d\theta \\ + (n-1)(\cot_K r) \int u^2 (\sin_K r)^{n-1} \sqrt{a_K}\, d\theta . \end{aligned}$$

Lemma 5.8.

$$\int 2uu_r (\sin_K r)^{n-1} \sqrt{a_K}\, d\theta \geq 0 .$$

Proof. By Green's formula and the harmonicity of u

$$\begin{aligned} \int 2uu_r (\sin_K r)^{n-1} \sqrt{a_K}\, d\theta &= \int_{\partial B(p,r)} \frac{\partial (u^2)}{\partial \hat{n}}\, dA_r \\ &= \int_{B(p,r)} \Delta(u^2)\, d\mathrm{Vol} = \int_{B(p,r)} 2|\nabla u|^2\, d\mathrm{Vol} . \end{aligned}$$

□

Proof of Theorem 2.3, part (i). Part (i) of the theorem follows directly from Lemma 5.7, Lemma 5.8 and Lemma 5.1. □

5.4 Proof of part (ii)

Let $w = (\sin_K r)^l u$, where $l = (n-2)/2$. w satisfies the equation

$$w_{rr} + (\cot_K r + \gamma_K)w_r + l(l+1)Kw - \frac{l^2 w}{(\sin_K r)^2} + \frac{\Delta_S w}{(\sin_K r)^2} = 0 \ . \tag{5.9}$$

Let

$$Q(r) = \int w(r,\theta)^2 \sqrt{a_K}\, d\theta = \frac{q(r)}{\sin_K(r)} \ . \tag{5.10}$$

Let us also set

$$\nabla_S w := (\sin_K r)\left(\nabla w - w_r \frac{\partial}{\partial r}\right) = \frac{1}{\sin_K r} a_K^{ij} \frac{\partial w}{\partial \theta^i} \frac{\partial}{\partial \theta^j} \ .$$

∇_S is defined in this way in order to have Green's formula

$$\int f(\theta)(\Delta_S g)(\theta)\sqrt{a_K}\, d\theta = -\int \langle \nabla_S f, \nabla_S g \rangle \sqrt{a_K}\, d\theta \ . \tag{5.11}$$

Note also that $\langle \nabla_S w, \partial_r \rangle = 0$.

Lemma 5.12. (i) $Q'(r) = \int 2w(w_r + \gamma_K w/2)\sqrt{a_K}\, d\theta$.

(ii) $Q'(r) \geq (n-2)(\cot_K r)Q(r) \geq 0$.

Proof. Part (i) is a direct calculation. Part (ii) is just another formulation of part (i) of Theorem 2.3. □

A second direct calculation using equation (5.9) and formula (5.11) gives

Lemma 5.13.

$$\begin{aligned} Q''(r) + (\cot_K r)Q'(r) &= 2\int \left(w_r + \frac{\gamma_K}{2} w\right)^2 \sqrt{a_K}\, d\theta \\ &+ \frac{2}{(\sin_K r)^2}\int |\nabla_S w|^2\, d\theta + \frac{2l^2}{(\sin_K r)^2}\int w^2 \sqrt{a_K}\, d\theta \\ &- 2l(l+1)K\int w^2\sqrt{a_K}\, d\theta + \int w^2 \left(\gamma_{K,r} + \gamma_K \cot_K r + \frac{\gamma_K^2}{2}\right)\sqrt{a_K}\, d\theta \ . \end{aligned}$$

Lemma 5.14.

$$Q''(r) + (\cot_K r)Q'(r) \geq 2 \int \left(w_r + \frac{\gamma_K}{2} w\right)^2 \sqrt{a_K}\, d\theta$$
$$+ \frac{2}{(\sin_K r)^2} \int |\nabla_S w|^2 \sqrt{a_K}\, d\theta + \frac{2l^2}{(\sin_K r)^2} Q - 2l(l+1)KQ - (n-1)(K-\kappa)Q\ .$$

Proof. This estimate is due to Lemma 5.13 and the estimates on γ_K and $\gamma_{K,r}$ in Lemma 5.1 and Lemma 5.2 respectively. ☐

Immediately we get

Lemma 5.15.

$$Q''(r) + (\cot_K r)Q'(r) - \frac{Q'(r)^2}{Q(r)} \geq 2 \int \left(w_r + \frac{\gamma_K}{2} w\right)^2 \sqrt{a_K}\, d\theta$$
$$+ \frac{2}{(\sin_K r)^2} \int |\nabla_S w|^2 \sqrt{a_K}\, d\theta + \frac{2l^2}{(\sin_K r)^2} Q - 2l(l+1)KQ - (n-1)(K-\kappa)Q$$
$$- \frac{4\left(\int w(w_r + \gamma_K w/2) \sqrt{a_K}\, d\theta\right)^2}{\int w^2 \sqrt{a_K}\, d\theta}$$

Lemma 5.16.

$$Q''(r) + (\cot_K r)Q'(r) - \frac{Q'(r)^2}{Q(r)} + (n-1)(\cot_\kappa r - \cot_K r)Q'(r)$$
$$\geq \frac{\varphi(r)}{(\sin_K r)^2} + \frac{2l^2}{(\sin_K r)^2} Q - 2l(l+1)KQ - (n-1)(K-\kappa)Q$$

where

$$\varphi(r) = -2(\sin_K r)^2 \int w_r^2 \sqrt{a_K}\, d\theta + 2 \int |\nabla_S w|^2 \sqrt{a_K}\, d\theta\ .$$

Proof.

$$\begin{aligned}
Q''(r) &+ (\cot_K r)Q'(r) - \frac{Q'(r)^2}{Q(r)} \\
&\geq 2\int \left(w_r + \frac{\gamma_K}{2}w\right)^2 \sqrt{a_K}\, d\theta \\
&+ \frac{2}{(\sin_K r)^2}\int |\nabla_S w|^2 \sqrt{a_K}\, d\theta + \frac{2l^2}{(\sin_K r)^2}Q \\
&- 2l(l+1)KQ - (n-1)(K-\kappa)Q \\
&- \frac{2\left(\int w(w_r + \gamma_K w/2)\sqrt{a_K}\, d\theta\right)^2}{\int w^2 \sqrt{a_K}\, d\theta} - \frac{2\left(\int w(w_r + \gamma_K w/2)\sqrt{a_K}\, d\theta\right)^2}{\int w^2 \sqrt{a_K}\, d\theta} \\
&\geq \frac{2}{(\sin_K r)^2}\int |\nabla_S w|^2 \sqrt{a_K}\, d\theta + \frac{2l^2}{(\sin_K r)^2}Q - 2l(l+1)KQ \\
&- (n-1)(K-\kappa)Q - \frac{2\left(\int w w_r \sqrt{a_K}\, d\theta + \int \gamma_K w^2/2 \sqrt{a_K}\, d\theta\right)^2}{\int w^2 \sqrt{a_K}\, d\theta} \\
&\geq \frac{\varphi(r)}{(\sin_K r)^2} + \frac{2l^2}{(\sin_K r)^2}Q - \frac{\int \gamma_K w^2 \sqrt{a_K}\, d\theta}{\int w^2 \sqrt{a_K}\, d\theta}Q' + \frac{(\int \gamma_K w^2 \sqrt{a_K}\, d\theta)^2}{2\int w^2 \sqrt{a_K}\, d\theta} \\
&- 2l(l+1)KQ - (n-1)(K-\kappa)Q \\
&\geq \frac{\varphi(r)}{(\sin_K r)^2} + \frac{2l^2}{(\sin_K r)^2}Q \\
&- (n-1)(\cot_\kappa r - \cot_K r)Q' - 2l(l+1)KQ - (n-1)(K-\kappa)Q\ .
\end{aligned}$$

The first inequality is just a rewriting of Lemma 5.15. In the second inequality we applied Cauchy-Schwarz inequality on the last term. In the third inequality we unfolded the parentheses in the last term and applied Cauchy-Schwarz inequality on the term $\int w w_r \sqrt{a_K}\, d\theta$. In the last inequality we used the fact that $Q' \geq 0$ (Lemma 5.12) and the estimates on γ_K in Lemma 5.1. ☐

It remains to control the function φ in terms of Q and Q'. We would like first to calculate the derivative of φ. To that end, we recall the definition and some of the properties of the Hessian as a bilinear form:

$$\mathrm{Hess} f(X,Y) := XYf - (\nabla_X Y)f = \langle Y, \nabla_X \mathrm{grad} f\rangle = \langle X, \nabla_Y \mathrm{grad} f\rangle\ .$$

In a geodesic ball centred at p, we have a radial field $\mathrm{grad}\, r = \partial_r$, tangent to the geodesics emanating from p. Since ∂_r is tangent to a geodesic, we have

$\nabla_{\partial_r}\partial_r = 0$. As a consequence $(\operatorname{Hess} r)(\partial_r, Y) = 0$ for all vectors Y. When computing the derivative of φ, it is convenient to have the following formula:

Lemma 5.17.

$$(|\nabla_S f|^2)_r = 2\langle \nabla_S f, \nabla_S f_r\rangle - 2\mathrm{Hess}(r)(\nabla_S f, \nabla_S f) + 2(\cot_K r)|\nabla_S f|^2$$

Proof.

$$\begin{aligned}
2\mathrm{Hess}(r)(\nabla_S f, \nabla_S f) &= 2(\sin_K r)^2 \mathrm{Hess}(r)(\nabla f, \nabla f) \\
&= 2(\sin_K r)^2 \langle \nabla f, \nabla_{\nabla f}\partial_r\rangle = 2(\sin_K r)^2 \langle \nabla f, \nabla_{\partial_r}\nabla f + [\nabla f, \partial_r]\rangle \\
&= (\sin_K r)^2 (|\nabla f|^2)_r + 2(\sin_K r)^2 [\nabla f, \partial_r] f \\
&= -(\sin_K r)^2 (|\nabla f|^2)_r + 2(\sin_K r)^2 \langle \nabla f, \nabla f_r\rangle \\
&= -(\sin_K r)^2 \left(f_r^2 + (\sin_K r)^{-2}|\nabla_S f|^2\right)_r + 2(\sin_K r)^2 f_r f_{rr} \\
&\quad + 2\langle \nabla_S f, \nabla_S f_r\rangle = -(|\nabla_S f|^2)_r + 2(\cot_K r)|\nabla_S f|^2 + 2\langle \nabla_S f, \nabla_S f_r\rangle
\end{aligned}$$

□

Using the formula in Lemma 5.17 we can readily compute the derivative of $\varphi(r)$ (defined in Lemma 5.16):

Lemma 5.18.

$$\begin{aligned}
\varphi'(r) = -4 &\int \mathrm{Hess}(r)(\nabla_S w, \nabla_S w)\, \sqrt{a_K}\, d\theta \\
&+ 4(\cot_K r) \int |\nabla_S w|^2 \sqrt{a_K}\, d\theta + 2l(l+1)K(\sin_K r)^2 Q' - 2l^2 Q' \\
&+ 2(\sin_K r)^2 \int |\nabla w|^2 \gamma_K \sqrt{a_K}\, d\theta \\
&+ 2l^2 \int w^2 \gamma_K \sqrt{a_K}\, d\theta - 2l(l+1)K \sin_K^2 r \int w^2 \gamma_K \sqrt{a_K}\, d\theta
\end{aligned}$$

Lemma 5.19.

$$\begin{aligned}
\varphi'(r) \geq -4(\cot_\kappa r - \cot_K r) &\int |\nabla_S w|^2 \sqrt{a_K}\, d\theta + 2l(l+1)K(\sin_K r)^2 Q' - 2l^2 Q' \\
&- 2l(\cot_\kappa r - \cot_K r)K^+(\sin_K r)^2 Q\ .
\end{aligned}$$

Proof. This is due to inequality (5.3) and Lemma 5.18. □

In Lemma 5.24 we integrate the inequality in Lemma 5.19. We need a few lemmas before that:

Lemma 5.20.

$$\left(1-\frac{2}{n}\right)(\sin_K r)^2 Q(r) \leq \int_0^r (\sin_K \rho)^2 Q'(\rho)\, d\rho \leq (\sin_K r)^2 Q(r)\ .$$

Proof. The RHS follows from the fact that $\sin_K \rho$ is monotonically increasing in ρ and $Q' \geq 0$. By derivating the LHS we see that it is enough to prove

$$\left(1-\frac{2}{n}\right)(\sin_K r)^2(2(\cot_K r)Q(r)+Q'(r)) \leq (\sin_K r)^2 Q'(r)\ . \tag{5.21}$$

Inequality (5.21) is equivalent to part (ii) of Lemma 5.12. $\square$

Lemma 5.22.

$$\int_0^r \int |\nabla_S w|^2 \sqrt{a_K}\, d\theta\, d\rho \leq \frac{(\sin_K r)^2}{2}\left(Q'(r)-(n-2)(\cot_K r)Q\right)\ .$$

Proof.

$$\begin{aligned}
\int_0^r \int |\nabla_S w|^2 \sqrt{a_K}\, d\theta\, d\rho &= \int_0^r \int |\nabla_S u|^2 (\sin_K \rho)^{n-2} \sqrt{a_K}\, d\theta\, d\rho \\
&\leq \int_0^r \int |\nabla u|^2 (\sin_K \rho)^{n} \sqrt{a_K}\, d\theta\, d\rho \\
&\leq \sin_K r \int_0^r \int |\nabla u|^2 (\sin_K \rho)^{n-1} \sqrt{a_K}\, d\theta\, d\rho \\
&= \sin_K r \int_{B(p,r)} |\nabla u|^2\, d\mathrm{Vol} = \sin_K r \int u u_r (\sin_K r)^{n-1} \sqrt{a_K}\, d\theta \\
&= (\sin_K r)^2 \int w w_r \sqrt{a_K}\, d\theta - l \cot_K r (\sin_K r)^2 \int w^2 \sqrt{a_K}\, d\theta \\
&= (\sin_K r)^2 \int w(w_r + \gamma_K w/2)\sqrt{a_K}\, d\theta - l(\cot_K r)(\sin_K r)^2 \int w^2 \sqrt{a_K}\, d\theta \\
&\quad - (\sin_K r)^2 \int w^2 \gamma_K/2 \sqrt{a_K}\, d\theta \leq \frac{(\sin_K r)^2}{2}\left(Q'(r)-(n-2)(\cot_K r)Q\right)\ .
\end{aligned}$$

$\square$

Lemma 5.23.

$$\int_0^r (\sin_K \rho)^2 Q(\rho)\ d\rho \le r(\sin_K r)^2 Q(r)\ .$$

Proof. $\sin_K \rho$ and $Q(\rho)$ are monotonically increasing in ρ. $\square$

Lemma 5.24.

$$\frac{\varphi(r)}{(\sin_K r)^2} \ge -2(\cot_\kappa r - \cot_K r)\,(Q' - (n-2)(\cot_K r)Q)$$
$$+\frac{n(n-2)}{2}KQ - (n-2)K^+Q - \frac{(n-2)^2 Q}{2(\sin_K r)^2} - (n-2)(\cot_\kappa r - \cot_K r)rK^+Q.$$

Proof. Observe that the functions $\cot_\kappa r - \cot_K r$ and $\sin_K r$ are both monotonically increasing. Hence, integrating Lemma 5.19, applying Lemmas 5.20–5.23 we obtain

$$\begin{aligned}\varphi(r) \ge -4(\cot_\kappa r - \cot_K r)\int_0^r \int |\nabla_S w|^2 \sqrt{a_K}\ d\theta\ d\rho \\ + 2l(l+1)K\int_0^r (\sin_K \rho)^2 Q'(\rho)\ d\rho - 2l^2 Q \\ - 2l(\cot_\kappa r - \cot_K r)K^+ \int_0^r (\sin_K \rho)^2 Q(\rho)\ d\rho \\ \ge -2(\sin_K r)^2(\cot_\kappa r - \cot_K r)\,(Q' - (n-2)(\cot_\kappa r - \cot_K r)(\cot_K r)Q) \\ + 2l(l+1)K(\sin_K r)^2 Q(r) - 2l(l+1)K^+(2/n)(\sin_K r)^2 Q - 2l^2 Q \\ - (n-2)(\cot_\kappa r - \cot_K r)rK^+(\sin_K r)^2 Q(r)\ .\end{aligned}$$

$\square$

Proof of Theorem 2.3, part (ii). From Lemma 5.16 and Lemma 5.24 we get

$$\begin{aligned}Q''(r) + (\cot_K r)Q'(r) - \frac{Q'(r)^2}{Q(r)} + (n-1)(\cot_\kappa r - \cot_K r)Q'(r) \\ \ge -2(\cot_\kappa r - \cot_K r)Q'(r) + 2(n-2)(\cot_\kappa r - \cot_K r)(\cot_K r)Q \\ +\frac{n(n-2)}{2}KQ(r) - (n-2)K^+Q - \frac{(n-2)^2}{2(\sin_K r)^2}Q - (n-2)(\cot_\kappa r - \cot_K r)rK^+Q \\ - \frac{n(n-2)}{2}KQ + \frac{(n-2)^2}{2(\sin_K r)^2}Q - (n-1)(K-\kappa)Q = \\ -2(\cot_\kappa r - \cot_K r)Q'(r) + 2(n-2)(\cot_\kappa r - \cot_K r)(\cot_K r)Q \\ -(n-2)K^+Q - (n-2)(\cot_\kappa r - \cot_K r)rK^+Q - (n-1)(K-\kappa)Q\ .\end{aligned} \tag{5.25}$$

We get

$$
\begin{aligned}
(\log Q)''(r) + (\cot_K r)(\log Q)'(r) + (n+1)(\cot_\kappa r - \cot_K r)(\log Q)'(r) \\
\geq -(n-1)(K-\kappa) - (n-2)K^+ - (n-2)(K-\kappa)\frac{r^2 K^+}{3} \\
\geq -(2n-3)(K-\kappa) - (n-2)K^+ \ , \quad (5.26)
\end{aligned}
$$

where we applied parts (i) and (ii) of Lemma 5.4. Recall $q(r) = Q(r)(\sin_K r)$. A direct computation shows

$$
\begin{aligned}
(\log \sin_K r)'' + (\cot_K r)(\log \sin_K r)' + (n+1)(\cot_\kappa r - \cot_K r)(\log \sin_K r)' \\
= -K + (n+1)\cot_K r(\cot_\kappa r - \cot_K r) \geq -K \ . \quad (5.27)
\end{aligned}
$$

Finally, adding up (5.26) and (5.27) gives the statement in the theorem. □

5.5 Proof of Corollary 2.4

Proof of Corollary 2.4. From Theorem 2.3

$$
q''(r) + (\cot_K r)q' + (n+1)(\cot_{-K} r - \cot_K r)q'(r) \geq -(5n-7)Kq \quad (5.28)
$$

From Lemma 5.4 and from the fact that $q' \geq 0$ (part (i) of Theorem 2.3) we know that

$$
(n+1)(\cot_{-K} r - \cot_K r)q' \leq 2(n+1)rKq'/3 \ . \quad (5.29)
$$

From part (i) of Theorem 2.3 we know that

$$
-(5n-7)Kq \geq -5Kq'(r)/\cot_K r \ . \quad (5.30)
$$

It is easy to check that $1/\cot_K r \leq 2r$ for $r \leq \pi/(3\sqrt{K})$.

Hence, from inequalities (5.28)–(5.30) we get

$$
q''(r) + \frac{1+8nr^2K}{r}q' - \frac{q'(r)^2}{q(r)} \geq 0 \quad (5.31)
$$

for $r\sqrt{K} < \pi/3$. If we define $l(t) = q(\mathrm{e}^t)$ then (5.31) is equivalent to

$$
l''(t) + 8nK\mathrm{e}^{2t}l'(t) \geq 0 \quad (5.32)
$$

for $t < -(\log K)/2 + \log(\pi/3)$. We will now integrate inequality (5.32).

Inequality (5.32) can be rewritten as $(\mathrm{e}^{4nK\mathrm{e}^{2t}} l'(t))' \geq 0$, from which we see that for $s_2 < s_1$

$$l'(s_2) \leq \mathrm{e}^{4nK(\mathrm{e}^{2s_1}-\mathrm{e}^{2s_2})} l'(s_1) \leq \mathrm{e}^{4nK\mathrm{e}^{2s_1}} l'(s_1) \ , \tag{5.33}$$

where, the last inequality is true since $l'(s) \geq 0$ from part (i) of Theorem 2.3. Hence for $t_2 < t_1$ such that $16nK\mathrm{e}^{2t_1} < 1$, and $0 \leq h \leq \log 2$

$$\begin{aligned} l(t_2 + h) - l(t_2) &= \int_0^h l'(t_2 + s)\, ds \leq \int_0^h \mathrm{e}^{4nK\mathrm{e}^{2t_1+2s}} l'(t_1 + s)\, ds \\ &\leq \mathrm{e}^{4nK\mathrm{e}^{2t_1+2h}} (l(t_1 + h) - l(t_1)) \leq (1 + 32nK\mathrm{e}^{2t_1})(l(t_1 + h) - l(t_1)) \ . \end{aligned}$$

The last inequality follows from $\mathrm{e}^x \leq 1 + 2x$ for $0 \leq x \leq 1$.

Going back from the variable t to the variable r we obtain the stated corollary. □

6 The case of constant curvature manifolds

We give a new proof of Theorem 2.1 and a second proof of Theorem 2.3 in the case of constant nonzero curvature in dimension two.

6.1 Zero curvature

Let $u_l(r, \theta) = r^l \cos(l\theta)$, $v_l = r^l \sin(l\theta)$. $q_{u_l}(r) = q_{v_l}(r) = \pi r^{2l+1}$. It is obvious that $\log q_l$ is a convex function of $\log r$.

Now, any harmonic function can be written as

$$u = a_0 + \sum_{l=1}^{\infty} a_l u_l(r, \theta) + b_l v_l(r, \theta) \ .$$

The functions $u_l(r, \cdot), v_l(r, \theta)$ are pairwise orthogonal as functions on the unit circle for all fixed r. For any two orthogonal functions f, g on the unit circle for all fixed r we have $q_{f+g}(r) = q_f(r) + q_g(r)$. We also know that the sum of log-convex functions is log-convex and the pointwise limit of log-convex functions is log-convex. These considerations give a short new proof of Theorem 2.1.

Remark. A similar argument carries out also in dimensions ≥ 3.

6.2 Positive curvature, dimension two

The metric on the 2-dimensional sphere of constant curvature $K > 0$ is given by
$$ds^2 = dr^2 + (\sin_K r)^2 \, d\theta^2 \ .$$
Here $0 \le r < \pi/\sqrt{K}$, and $0 \le \theta \le 2\pi$. Hence,
$$q_u^K(r) = \int_0^{2\pi} u(r,\theta)^2 (\sin_K r) \, d\theta.$$
We define also $q_f^0(r) = \int_0^{2\pi} f(r,\theta)^2 r \ d\theta$ for function defined on $\mathbb{R}^2$.

Let $f(r,\theta)$ be defined on $\mathbb{R}^2$ by $u(r,\theta) = f(\tan(r\sqrt{K}/2), \theta)$. f is related to u by a stereographic projection. Since harmonic functions are preserved under conformal transformations in dimension two, $f(r,\theta)$ is harmonic if and only if $u(r,\theta)$ is harmonic. We also note the relation
$$q_u^K(r) = \frac{q_f^0(\tan(r\sqrt{K}/2))}{\tan(r\sqrt{K}/2)} \sin_K r \ .$$

Suppose now f is harmonic. Then, from the fact that $\log q_f^0$ is a convex function of $\log r$, we obtain

Theorem 6.1. *If $K > 0$ then*
$$(\log q_u^K)''(r) + (\cot_K r)(\log q_u^K)'(r) \ge -K \ .$$

6.3 Negative curvature

In the spherical example one can replace all trigonometric functions by the corresponding hyperbolic functions and obtain

Theorem 6.2. *If $K < 0$ then*
$$(\log q_u^K)''(r) + (\cot_K r)(\log q_u^K)'(r) \ge -K > 0 \ .$$

7 Discussion

We raise several questions which we find interesting to pursue.

7.1 Beyond the injectivity radius

It would be interesting to understand whether Theorem 2.3 remains true beyond the injectivity radius as long as $r\sqrt{K^+} < \pi/2$ in the spirit of Bishop-Gromov's Volume Comparison Theorem ([Gro81]).

7.2 Proof by an orthogonal basis of functions.

In a manifold of constant curvature $K \neq 0$ of dimension ≥ 3 we would like to have a simple proof, inspired from the proof presented in section 6 for the case $K = 0$. This would shed light also on the sharpness of Theorem 2.3 in dimensions $n \geq 3$.

7.3 Ricci curvature.

Can one of the bound assumptions on the sectional curvature in Theorem 2.3 be relaxed to a bound on the Ricci curvature?

7.4 Eigenfunctions on negatively curved manifolds.

Can we replace the extension procedure described in Section 3 by a procedure which will give us more information on the growth of eigenfunctions on negatively curved manifolds?

7.5 A comparison theorem for positive harmonic functions

Let $f(\theta)$ be a 2π-periodic non-negative function. Let u be a solution of the Dirichlet problem in the unit disk: $\Delta u = 0$ with $u(1, \theta) = f(\theta)$. Now, suppose we consider the unit geodesic disk in a Riemannian manifold with non-positive variable curvature, and solve the Dirichlet problem there. We get a solution $v(r, \theta)$. Can we compare the values of u to the values of v? Or equivalently, can we compare the Poisson kernels involved?

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